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Loop Equations for + and − Loops in $c = \frac{1}{2}$ Non-Critical String Theory

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Abstract

New loop equations for all genera in $c = \frac{1}{2}$ non-critical string theory are constructed. Our loop equations include two types of loops, loops with all Ising spins up (+ loops) and those with all spins down (− loops). The loop equations generate an algebra which is a certain extension of W_3 algebra and are equivalent to the W_3 constraints derived before in the matrix-model formulation of 2d gravity. Application of these loop equations to construction of Hamiltonian for $c = \frac{1}{2}$ string field theory is considered.

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1 Introduction

Loop equations frequently appear in 2d quantum gravity. For example, they were used to show that one-matrix model has various critical points that correspond to non-unitary conformal fields coupled to 2d gravity. [1] In the case of (p,q) conformal fields coupled to 2d gravity it was shown that loop equations can be written in the form of W_p constraints on generating function of correlation functions for scaling operators.[2][3] Loop equations in other models such as ADE models, O(n) models on a random lattice have also been studied.[4]

String field theory (SFT)[5] is important for nonperturvative study of string theories. Several years ago a new type of SFT was considered in [6] for $c = 0$ noncritical string (2d pure gravity). Much effort has been devoted to extending this formalism to other $c \leq 1$ strings.[7]$\sim$[13] Satisfactory formalism, however, seems to be still elusive.

In the study of SFT it was realized that hamiltonian in SFT is in close relationship with loop equations.[8] Therefore investigation of loop equations is an important step toward construction of SFT. In this paper we will concentrate on $c = 1/2$ string, *i.e.* continuum limit of Ising model coupled to 2d gravity. In this theory many types of boundary conditions for Ising spins can be considered. Apparently it seems that by restricting the spin configurations on the loops differently we obtain different versions of SFT.[7][8][9][13] Here we will consider only two types of spin configurations on loops; all Ising spins on loops are either up or down. Throughout this paper we will call a loop with all spins on it up a $+$ loop, and a loop with all spins down a $-$ loop. The main purpose of this paper is to construct loop equations in $c = 1/2$ string for $+$ and $-$ loops. Such loop equations should be constructed in such a way that they are equivalent to W_3 constraints [2][3][14] derived in the matrix model formulation [15] $\sim$ [17] of 2d gravity.

$c = 1/2$ SFT for the above boundary conditions has already been presented in [7]. In this reference they assumed a certain Hamiltonian for $c = 1/2$ SFT from the outset and derived loop equations as Schwinger-Dyson equations (SDE) in SFT. Their loop equations, however, turned out to be a set of two decoupled Virasoro constraints and the connection of their loop equations with W_3 constraints is not clear. The purpose of the present work is to derive loop equations which are directly related to W_3 constraints.

The loop operators $w_+(l)$, $w_-(l)$ which create $+$ and $-$ loops of length l can be formally expanded in terms of the scaling operators. [17][18] Therefore we can expect that if we consider only one type of loops (*e.g.* $+$ loops), the

structure of loop equations will be very similar to that of W_3 constraints in matrix models. Indeed loop equations for + loops can be derived in this way. It turns out that two independent source functions $J_+^{(1)}(l)$, $J_+^{(2)}(l)$ have to be introduced for a + loop of length l. However, it is rather intricate to include $-$ loops into the loop equations. We will do this by using the relationship between $w_+(l)$ and $w_-(l)$ through analytic continuation in l. We will show that in this procedure we have to carefully treat the singular terms in two-loop amplitudes. As we will see later the structure of the resultant loop equations is very intriguing.

In sec 2 we will write down the loop equations for + loops and show these equations satisfy consistency conditions. The differential operators in the loop equations generate the continuum version of W_3 algebra. In sec 3 we derive a relationship between + and $-$ loops in terms of analytic continuation in the length variable l of loops. We point out the problem which arises in Laplace transformation of this relation and derive correct formulae. By using these formulae the loop equations for + and $-$ loops are obtained in sec 4. In sec 5 we will show that these loop equations satisfy the consistency conditions. These loop equations generate a generalization of W_3 algebra. An interesting structure of the loop equations (*e.g.* $T_+ - V_-$ followed by Bogoliubov transformation) is noticed. In sec 6 we will consider SFT as an application of our loop equations. The problem of consistency condition on string field Hamiltonian is discussed. In sec 7 we will give a brief summary and discussions. A reduced version of the loop equations is also derived by eliminating extra degrees of freedom $\bar{J}_\pm = (J_\pm^{(1)} - J_\pm^{(2)})/2$. In appendix A one of the loop equations $lU_+(l)Z_{+-} = 0$ obtained in sec 4 is shown to be equivalent to the Virasoro constraint in matrix models. In appendix B an explicit form of the W_3 current $X_+(l)$ defined in sec 5 is presented and the algebra that $U_\pm$ and $X_\pm$ generate is displayed. Some of the results in this paper were presented in [19].

2 Loop Equations for + Loops

A generating function for Green functions of scaling operators $\mathcal{O}_n$ ($n \neq 0$ (mod 3)) in $c = 1/2$ string theory,

$$\begin{aligned}\tau(\mu) &= \tau(\mu_1, \mu_2, \mu_4, \mu_5, \mu_7, \mu_8, \cdots) \\ &= < \exp\{\sum_{n=0}^{\infty}(\mu_{3n+1}\mathcal{O}_{3n+1} + \mu_{3n+2}\mathcal{O}_{3n+2})\} > \end{aligned} \tag{2.1}$$

satisfies the W_3 constraints

$$\begin{aligned} L_n\tau(\mu) &= 0 \qquad n=-1,0,1,\cdots && (2.2)\\ W_n\tau(\mu) &= 0 \qquad n=-2,-1,0,1,\cdots, && (2.3)\end{aligned}$$

where L_n and W_n are differential operators with respect to μ's, [1] which generate the W_3 algebra [20]

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{1}{12}n(n^2-1)\delta_{n,-m}, \tag{2.4}$$

$$[L_n,W_m]=(2n-m)W_{n+m}, \tag{2.5}$$

$$\begin{aligned}[][W_n,W_m] &= -9(n-m)U_{n+m}-\frac{1}{10}n(n^2-1)(n^2-4)\delta_{n,-m}\\ &\quad +(n-m)\{\frac{3}{2}(n^2+4nm+m^2)+\frac{27}{2}(n+m)+21\}L_{n+m}.\\ &\quad (U_n=\sum_{k\leq -2}L_kL_{n-k}+\sum_{k\geq -1}L_{n-k}L_k) && (2.6)\end{aligned}$$

These results for $c=1/2$ string were first conjectured in the study of SDE for large-N matrix models [2] [3] and confirmed in [14]. These L_n's and W_n's can be succinctly expressed in terms of a complex Z_3-twisted scalar field $\phi(z)$

$$\begin{aligned} T(z) &= -:\partial_z\phi^*(z)\partial_z\phi(z):+\frac{1}{9z^2}\equiv\sum_n z^{-n-2}L_n,\\ W(z) &= (\partial_z\phi(z))^3+(\partial_z\phi^*(z))^3\equiv\sum_n z^{-n-3}W_n, && (2.7)\end{aligned}$$

where the fields $\phi(z)$, $\phi^*(z)$ have the following mode expansions [2]

$$\begin{aligned} \phi(z) &= \sum_{n=0}^{\infty}(z^{n+\frac{1}{3}}\frac{\mu_{3n+1}}{\sqrt{g}}-z^{-n-\frac{2}{3}}\frac{3\sqrt{g}}{3n+2}\frac{\partial}{\partial\mu_{3n+2}}),\\ \phi^*(z) &= \sum_{n=0}^{\infty}(-z^{n+\frac{2}{3}}\frac{\mu_{3n+2}}{\sqrt{g}}+z^{-n-\frac{1}{3}}\frac{3\sqrt{g}}{3n+1}\frac{\partial}{\partial\mu_{3n+1}}). && (2.8)\end{aligned}$$

Here g is a string-coupling constant.

The relationship between the SDE for matrix models and the loop equations is well known [6][7][8][9]. The functional differential operaters appearing in loop equations generate a 'continuum' Virasoro algebra.[7] From the above results we are naturally led to introduce two independent source functions

[1] See eqs(2.7),(2.8) below

[2] Normal ordering $:\cdots:$ is defined by regarding μ as an annihilation operator and $\partial/\partial\mu$ a creation operator.

$J_+^{(1)}(l), J_+^{(2)}(l)$ to construct loop equations for + loops. Actually it is not difficult to figure out that loop equations take the following forms

$$\begin{aligned} lT_+(l)Z_+[J_+^{(1)}, J_+^{(2)}] &= 0, & (2.9)\\ l^2 W_+(l)Z_+[J_+^{(1)}, J_+^{(2)}] &= 32\{6\delta''(l) - t\delta(l)\}Z_+[J_+^{(1)}, J_+^{(2)}] & (2.10)\end{aligned}$$

Here Z_+ is a generating function for amplitudes of + loops and $T_+(l)$, $W_+(l)$ are given by

$$T_+(l) = \{D_+^{(1)} * D_+^{(2)} + g\sum_{r=1}^{2}(lJ_+^{(r)}) \triangleleft D_+^{(r)}\}(l), \tag{2.11}$$

$$\begin{aligned} W_+(l) &= \sum_{r=1}^{2}\{D_+^{(r)} * D_+^{(r)} * D_+^{(r)} + 3g(lJ_+^{(3-r)}) \triangleleft (D_+^{(r)} * D_+^{(r)}) \\ &\quad +3g^2(lJ_+^{(3-r)}) \triangleleft ((lJ_+^{(3-r)}) \triangleleft D_+^{(r)})\}(l) \end{aligned} \tag{2.12}$$

Here $D_+^{(r)}(l)$ stands for $\delta/\delta J_+^{(r)}(l)$. The symbols $*$ and $\triangleleft$ represent the following integrals.[8]

$$(f * g)(l) \equiv \int_0^l dl' f(l')g(l-l'), \qquad (f \triangleleft g)(l) \equiv \int_0^\infty dl' f(l')g(l'+l). \tag{2.13}$$

Later we will also need the integral

$$(f \triangleright g)(l) \equiv \int_0^\infty dl f(l'+l)g(l') \tag{2.14}$$

It has to be stressed that $J_+^{(1)}$ and $J_+^{(2)}$ in the above formulae are not the source functions for the loop of length l with a fixed boundary condition (+ spins). Only the linear combination $J_+(l) = (J_+^{(1)}(l) + J_+^{(2)}(l))/2$ gives the source function for such a loop. J_+ is decomposed into $J_+^{(1)}$ and $J_+^{(2)}$ according to the powers of l contained.

$$J_+^{(1)}(l) \sim l^{-1/3-n}, \quad J_+^{(2)}(l) \sim l^{-2/3-n} \quad (n \in \mathbf{Z}) \tag{2.15}$$

For example, the disk amplitudes

$$f_+^{(r)}(l) = D_+^{(r)}(l) \ln Z|_{J=0,g=0} \tag{2.16}$$

are given [21] in laplace transformed forms [3] by

$$\begin{aligned} \tilde{f}_+^{(1)}(\zeta) &= \int_0^\infty dl e^{-\zeta l} f_+^{(1)}(l) = (\zeta - \sqrt{\zeta^2 - t})^{4/3}, \\ \tilde{f}_+^{(2)}(\zeta) &= \int_0^\infty dl e^{-\zeta l} f_+^{(2)}(l) = (\zeta + \sqrt{\zeta^2 - t})^{4/3} \end{aligned} \tag{2.17}$$

[3] Henceforth ~ will denote laplace transform

Here t is a cosmological constant.

It is not difficult to show that the disk amplitudes (2.17) satisfy the loop equations (2.9),(2.10). Conversely the loop equations determine the disk amplitudes completely. To show that these are actually equivalent to the W_3 constraints (2.2),(2.3) we have to factor out Z_+ into a singular piece Z_+^{sing} and a regular piece Z_+^{reg} as in the $c=0$ case. [6] We can show after a certain amount of calculation that

$$\tau(\mu) = Z_+^{reg}[J_+^{(1)}(l) + c_1 t l^{-4/3} + c_2 l^{-7/3}, J_+^{(2)}(l)] \tag{2.18}$$

with c_1, c_2 some constants satisfies W_3 constraints (2.2),(2.3).

Before we consider $-$ loops, we will comment on the algebra of $T_+(l)$ and $W_+(l)$. By explicit calculation we can show that these generate the 'continuum' W_3 algebra

$$[T_+(l), T_+(l')] = g(l-l')T_+(l+l'), \tag{2.19}$$

$$[T_+(l), W_+(l')] = g(2l-l')W(l+l'), \tag{2.20}$$

$$\begin{aligned}[W_+(l), W_+(l')] \;\; &= \;\; 9g(l-l')(T_+ * T_+)(l+l') + 18g(l-l')(V_+ \triangleleft T_+)(l+l') \\ &\quad -\frac{3}{2}g^2(l-l')(l^2+4ll'+l'^2)T_+(l+l'). \end{aligned} \tag{2.21}$$

Here $V_+(l)$ is defined by

$$V_+(l) = \{g\sum_{r=1}^{2}(lJ_+^{(r)}) \triangleright D_+^{(r)} + g^2(lJ_+^{(1)}) * (lJ_+^{(2)})\}(l) \tag{2.22}$$

and corresponds to Virasoro generators L_n with $n \leq -2$, while $T_+(l)$ to L_n with $n \geq -1$: we have

$$[T_+(l), V_+(l')] = g(l+l')T_+(l-l')\theta(l-l') + g(l+l')V_+(l'-l)\theta(l'-l), \tag{2.23}$$

$$[V_+(l), V_+(l')] = -g(l-l')V_+(l+l'), \tag{2.24}$$

where θ is a step function.

The algebra (2.19) $\sim$ (2.21) is not strictly closed because V_+ appears on the right-hand side of (2.21). Nonetheless this is sufficient for consistency of the loop equations (2.9), (2.10). [4] We can also show as in [7] that the right-hand side of (2.10) does not make (2.9),(2.10) inconsistent.

[4] Strictly speaking, for the proof of consistency we need more information than (2.9). Actually by explicitly solving (2.9) and (2.10) for Z_+ as perturbation series in g we can show that $T_+(l)Z_+ = -g\delta(l)\frac{\partial}{\partial t}Z_+$. A similar equation was also noticed in [13]

3 Relation between Loop Amplitudes for $+$ Loops and $-$ Loops

In this section we will derive formulae which relate $+$ and $-$ loops by analytic continuation in variables l, ζ. In the next section by using these formulae we will derive loop equations for both types of loops, $+$ and $-$ loops.

Let us denote the loop operators which create a loop of length l with $+$ spins and that with $-$ spins by $w_+(l)$ and $w_-(l)$, respectively. These are related by

$$w_-(l) = -w_+(e^{3\pi i} l) \tag{3.1}$$

This relation is implicit in the representation for loop amplitudes in terms of heat kernels.[17][22] For example one- and two-loop amplitudes can be written as

$$< w_\pm(l) > \propto \int_t^\infty dx < x | e^{\pm lL} | x >, \tag{3.2}$$

$$< w_+(l_1) w_\pm(l_2) >_c \propto \int_t^\infty dx \int_{-\infty}^t dy < x | e^{l_1 L} | y >< y | e^{\pm l_2 L} | x > \tag{3.3}$$

Here L is a third-order Lax operator given by

$$L = \frac{1}{2}\{-(\frac{d}{dx})^3 + 3x^{1/3}\frac{d}{dx}\} \tag{3.4}$$

These representations can be obtained by computing correlation functions in two-matrix model [16] by using the method of orthogonal polynomials and taking continuum limits of them.[5] A $-$ loop of length l appears as a $+$ loop of 'length $-l$' in these representations.

The reason for an extra minus sign on the right-hand side of (3.1) can be understood if we expand $w_\pm(l)$ in powers of l.

$$\begin{aligned} w_+(l) &= \sum_{n=0}^\infty \mathcal{O}_n^{(1)} l^{n+1/3} + \sum_{n=0}^\infty \mathcal{O}_n^{(2)} l^{n+2/3}, \\ w_-(l) &= \sum_{n=0}^\infty \mathcal{O}_n^{(1)} (-1)^n l^{n+1/3} - \sum_{n=0}^\infty (-1)^n \mathcal{O}_n^{(2)} l^{n+2/3} \end{aligned} \tag{3.5}$$

Here $\mathcal{O}_{3n+1} \equiv \mathcal{O}_n^{(1)}$ and $\mathcal{O}_{3n+2} \equiv \mathcal{O}_n^{(2)}$ are scaling operators. $\mathcal{O}_{2n}^{(1)}$ and $\mathcal{O}_{2n+1}^{(2)}$ are Z_2-even operators, while $\mathcal{O}_{2n+1}^{(1)}$ and $\mathcal{O}_{2n}^{(2)}$ are odd. We have $\exp\{3\pi i\}$ instead of $\exp\{\pi i\}$ in (3.1), because loop amplitudes have to be real functions.

[5] Loop amplitudes in (p,q) gravity were computed in [23]

To derive loop equations we need to laplace transform the relation (3.1). As far as loop amplitudes can be expanded in fractional powers of l as in (3.5), we can easily show that

$$\tilde{w}_-(\zeta) = \tilde{w}_+(e^{3\pi i}\zeta) \tag{3.6}$$

This relation (3.6) is valid for n-loop amplitudes with $n \geq 3$. In the case of one- and two-loop amplitudes we have to be careful in the treatment of singular terms. One-loop amplitudes can be expanded in powers of l, although first few terms are singular. Laplace transformation of these singular terms can be performed as usual in the sense of analytic continuation and the relation (3.6) is still valid. On the other hand two-loop amplitudes contain the singular terms which are not expandable.

$$< w_+(l_1)w_\pm(l_2) >_{c,sing} = \frac{\sqrt{3}}{2\pi} g \frac{(l_1 l_2)^{1/3}}{l_1 \pm l_2}(l_1^{1/3} \pm l_2^{1/3}) \tag{3.7}$$

The laplace transforms of these are given by

$$< \tilde{w}_+(\zeta_1)\tilde{w}_+(\zeta_2) >_{c,sing} = g\frac{(\zeta_1/\zeta_2)^{2/3} + 2(\zeta_1/\zeta_2)^{1/3} + 2(\zeta_2/\zeta_1)^{1/3} + (\zeta_2/\zeta_1)^{2/3} - 6}{3(\zeta_1 - \zeta_2)^2}, \tag{3.8}$$

$$< \tilde{w}_+(\zeta_1)\tilde{w}_-(\zeta_2) >_{c,sing} = g\frac{(\zeta_1/\zeta_2)^{2/3} - 2(\zeta_1/\zeta_2)^{1/3} - 2(\zeta_2/\zeta_1)^{1/3} + (\zeta_2/\zeta_1)^{2/3} + 3}{3(\zeta_1 + \zeta_2)^2}. \tag{3.9}$$

These yield the following relation

$$< \tilde{w}_+(\zeta_1)\tilde{w}_-(\zeta_2) >_c = < \tilde{w}_+(\zeta_1)\tilde{w}_+(e^{3\pi i}\zeta_2) >_c + \frac{3g}{(\zeta_1 + \zeta_2)^2} \tag{3.10}$$

We should also decompose the loop operators $w_+(l)$, $w_-(l)$ according to the powers of l.

$$w_\pm(l) = w_\pm^{(1)}(l) + w_\pm^{(2)}(l), \tag{3.11}$$

$$w_\pm^{(1)}(l) \sim l^{1/3+n}, \quad w_\pm^{(2)}(l) \sim l^{2/3+n}, \quad (n \in \mathbf{Z}) \tag{3.12}$$

It is natural to put the second term on the right-hand side of (3.10) into the relation for $< \tilde{w}^{(1)}\tilde{w}^{(2)} >$ and we obtain the following relations.

$$\begin{aligned} &< \tilde{w}_+^{(r_1)}(\zeta_1) \cdots \tilde{w}_+^{(r_n)}(\zeta_n)\tilde{w}_-^{(s_1)}(\xi_1) \cdots \tilde{w}_-^{(s_m)}(\xi_m) >_c \\ &= < \tilde{w}_+^{(r_1)}(\zeta_1) \cdots \tilde{w}_+^{(r_n)}(\zeta_n)\tilde{w}_+^{(s_1)}(e^{3\pi i}\xi_1) \cdots \tilde{w}_+^{(s_m)}(e^{3\pi i}\xi_m) >_c, \end{aligned}$$

$$< \tilde{w}_+^{(r)}(\zeta)\tilde{w}_-^{(s)}(\xi) >_c = < \tilde{w}_+^{(r)}(\zeta)\tilde{w}_+^{(s)}(e^{3\pi i}\xi) >_c + \frac{3}{2} g \frac{\delta_{r+s,3}}{(\zeta + \xi)^2} \tag{3.13}$$

Here $n + m \neq 2$ and $r, s, r_i, s_i = 1, 2$. These formulae will play an important role in the next section.

4 Inclusion of $-$ Loops into Loop Equations

We will derive the loop equations for loops with both types of spin configurations. For that purpose we have to introduce two more source functions $J_-^{(1)}(l)$, $J_-^{(2)}(l)$ for $-$ loops. We start with

$$D_+^{(1)}(l_2)\frac{1}{Z_+}l_1T_+(l_1)Z_+ = 0 \tag{4.1}$$

and set $J_+^{(1)} = J_+^{(2)} = 0$. By laplace transforming this ($\int_0^\infty dl_1 \int_0^\infty dl_2 \exp\{-\zeta_1 l_1 - \zeta_2 l_2\}$) and rewriting this in terms of loop operators, we obtain

$$\begin{aligned} -\tfrac{\partial}{\partial\zeta_1}[\;& < \tilde{w}_+^{(1)}(\zeta_1)\tilde{w}_+^{(2)}(\zeta_1)\tilde{w}_+^{(1)}(\zeta_2) >_c + < \tilde{w}_+^{(1)}(\zeta_1)\tilde{w}_+^{(1)}(\zeta_2) >_c < \tilde{w}_+^{(2)}(\zeta_1) > \\ & + < \tilde{w}_+^{(1)}(\zeta_1) >< \tilde{w}_+^{(2)}(\zeta_1)\tilde{w}_+^{(1)}(\zeta_2) >_c \\ & + g\frac{\partial}{\partial\zeta_2}\{\frac{1}{\zeta_1-\zeta_2}(< \tilde{w}_+^{(1)}(\zeta_1) > - < \tilde{w}_+^{(1)}(\zeta_2) >)\}] = 0. \end{aligned} \tag{4.2}$$

Here we replace ζ_2 by $e^{3\pi i}\zeta_2$ and use the relation (3.13) to rewrite the above equation. This yields

$$\begin{aligned} -\tfrac{\partial}{\partial\zeta_1}[\;& < \tilde{w}_+^{(1)}(\zeta_1)\tilde{w}_+^{(2)}(\zeta_1)\tilde{w}_-^{(1)}(\zeta_2) >_c + < \tilde{w}_+^{(1)}(\zeta_1)\tilde{w}_-^{(1)}(\zeta_2) >_c < \tilde{w}_+^{(2)}(\zeta_1) > \\ & + < \tilde{w}_+^{(1)}(\zeta_1) >< \tilde{w}_+^{(2)}(\zeta_1)\tilde{w}_-^{(1)}(\zeta_2) >_c \\ & + g\frac{\partial}{\partial\zeta_2}\{\frac{1}{\zeta_1+\zeta_2}(\frac{1}{2} < \tilde{w}_+^{(1)}(\zeta_1) > + < \tilde{w}_-^{(1)}(\zeta_2) >)\}] = 0. \end{aligned} \tag{4.3}$$

This equation can be furthermore rewritten as follows

$$\begin{aligned} D_-^{(1)}(l_2)\frac{l_1}{Z_{+-}}[&\int_0^{l_1} dl' D_+^{(1)}(l')D_+^{(2)}(l_1-l') - \frac{1}{2}g\int_0^{l_1} dl' l' J_-^{(1)}(l')D_+^{(1)}(l_1-l') \\ & -g\int_0^\infty dl'(l_1+l')J_-^{(1)}(l'+l_1)D_-^{(1)}(l')]Z_{+-}|_{J_+^{(r)}=J_-^{(r)}=0} = 0. \end{aligned} \tag{4.4}$$

Here $Z_{+-} = Z_{+-}[J_+^{(1)}, J_+^{(2)}, J_-^{(1)}, J_-^{(2)}]$ is a generating function for both types of loops.

By operating $D_+^{(r)}$ on eq (4.1) arbitrary times and repeating this procedure we finally arrive at the new loop equation

$$lU_+(l)Z_{+-} = 0, \tag{4.5}$$

$$lU_-(l)Z_{+-} = 0, \tag{4.6}$$

where $U_\pm(l)$ is the following operator.

$$U_\pm(l) \;=\; \{D_\pm^{(1)} * D_\pm^{(2)} + g\sum_{r=1}^{2}(lJ_\pm^{(r)}) \triangleleft D_\pm^{(r)} - g\sum_{r=1}^{2}(lJ_\mp^{(r)}) \triangleright D_\mp^{(r)}$$

$$+\alpha g\sum_{r=1}^{2}(lJ_{\mp}^{(r)})*D_{\pm}^{(r)}+(\alpha^2-1)g^2(lJ_{\mp}^{(1)})*(lJ_{\mp}^{(2)})\}(l),$$
$$(\alpha=-\frac{1}{2}). \tag{4.7}$$

Eq (4.6) is obtained from eq (4.5) by Z_2 symmetry for Ising spins. Let us note that if we set $J_{-}^{(1)}=J_{-}^{(2)}=0$, the loop equations (4.5), (4.6) reduce to those for only + loops (2.9).

Although we have derived loop equation (4.5) by using the method of analytical continuation (3.13), more direct proof will be desirable. In Appendix A we will prove that loop equation (4.5) is equivalent to the Virasoro constraint (2.2). We also checked that disk and cylinder amplitudes satisfy (4.5).

Let us make a remark on the value of α. We might formally repeat the procedure after (4.1) without laplace trasforming the loop equations. Because the singular parts of two-loop amplitudes (3.8), (3.9) respect the relation (3.1), this relation might be used for any amplitudes. In this case we would again end up with the loop equations (4.5), (4.6) but the value of α in (4.7) would then be replaced by 1. This discrepancy can be resolved if we take into account the fact that the $\int dl$ integrals appearing in the loop equations are divergent, when the lengths l of loops in one-loop amplitudes vanish, and we should not make the analytic continuation (3.1) in such divergent integrals. To the contrary Laplace-transformed loop equations like (4.2) are algebraic and cause no problems. Hence we should put $\alpha=-\frac{1}{2}$ as above.

5 Consistency Conditions

In this section we will show that $U_+(l)$ and $U_-(l)$ defined in the previous section satisfy consistency conditions. Let us define the operators

$$\hat{U}_+(l)\equiv T_+(l)-V_-(l),\qquad \hat{U}_-(l)\equiv T_-(l)-V_+(l), \tag{5.1}$$

where $T_-(l)$ and $V_-(l)$ are obtained from $T_+(l)$ and $V_+(l)$, respectively, by interchanging $J_+^{(r)}$ and $D_+^{(r)}$ with $J_-^{(r)}$ and $D_-^{(r)}$. Because $J_+^{(r)}$ and $J_-^{(r)}$ are independent functions, we can show by using the commutation relations (2.19), (2.23),(2.24) that $U_\pm(l)$ generate the algebra

$$[\hat{U}_\pm(l_1),\hat{U}_\pm(l_2)]=g(l_1-l_2)\hat{U}_\pm(l_1+l_2), \tag{5.2}$$

$$\begin{aligned}[\hat{U}_+(l_1),\hat{U}_-(l_2)] &= -g(l_1+l_2)\hat{U}_+(l_1-l_2)\theta(l_1-l_2)\\ &\quad +g(l_1+l_2)\hat{U}_-(l_2-l_1)\theta(l_2-l_1).\end{aligned} \tag{5.3}$$

These two generators can be combined into a single one: the operator

$$\hat{U}(l) \equiv \hat{U}_+(l)\theta(l) - \hat{U}_-(-l)\theta(-l) \qquad (-\infty < l < \infty) \tag{5.4}$$

generates a whole 'continuum' Virasoro algebra.

The functional differential operators $U_+(l)$, $U_-(l)$ which appear in loop equations (4.5), (4.6) turn out to be related to $\hat{U}_+(l)$ and $\hat{U}_-(l)$ by a Bogoliubov transformation

$$\begin{aligned} D_\pm^{(r)}(l) &\rightarrow D_\pm^{(r)}(l) - \frac{1}{2} g l J_\mp^{(3-r)}(l), \\ J_\pm^{(r)}(l) &\rightarrow J_\pm^{(r)}(l) \qquad (r = 1, 2) \end{aligned} \tag{5.5}$$

Because this transformation does not change the algebra, $U_+(l)$ and $U_-(l)$ generate two *coupled* Virasoro algebras (5.2),(5.3), and consistency conditions are satisfied. At present the meaning of this transformation is not clear to us.

The analysis in the preceding and present sections can be generalized to $W_+(l)$. We define

$$\begin{aligned} Y_-(l) = \quad & \sum_{r=1}^{2} [3g((lJ_-^{(r)}) \triangleright D_-^{(3-r)}) \triangleright D_-^{(3-r)} + 3g^2((lJ_-^{(r)}) * (lJ_-^{(r)})) \triangleright D_-^{(3-r)} \\ & + g^3 (lJ_-^{(r)}) * (lJ_-^{(r)}) * (lJ_-^{(r)})](l), \end{aligned} \tag{5.6}$$

and construct

$$\hat{X}_+(l) \equiv W_+(l) - Y_-(l). \tag{5.7}$$

By applying transformation (5.5) on $\hat{X}_+(l)$, we obtain $X_+(l)$. We will present it in Appendix B explicitly. Similarly we can construct $X_-(l)$. The operator $X_\pm(l)$ together with $U_\pm$ generate a closed algebra. This is also presented in Appendix B.

To summarize, loop equations for $c = 1/2$ string are given by [6]

$$\begin{aligned} & lU_+(l) Z_{+-} = lU_-(l) Z_{+-} = 0, \\ & l^2 X_+(l) Z_{+-} = l^2 X_-(l) Z_{+-} = 32(6\delta''(l) - t\delta(l)) Z_{+-} \end{aligned} \tag{5.8}$$

This is the main result of this paper.

[6] We can also show that $U_\pm(l) Z_{+-} = -g\delta(l) \frac{\partial}{\partial t} Z_{+-}$ by starting from the equation in footnote 4 and applying the same method as in sec 4

6 String Field Theory for $c = 1/2$ String

The loop equations obtained in the previous section determine the loop amplitudes completely. In this section we will discuss applications of these loop equations to string field theory.

Let us first consider a theory which contains only loops with + spins. In the string field theory we consider here[6], the generating function of loop amplitudes will be expressed in the form

$$Z_+[J_+^{(1)}, J_+^{(2)}] = \lim_{D\to\infty} < 0|e^{-DH} \exp\{\int_0^\infty dl \sum_{r=1,2} J_+^{(r)}(l)\Psi_+^{(r)\dagger}(l)\}|0 > \tag{6.1}$$

The parameter D may be interpreted as geodesic distance on the world sheet. Here $\Psi_+^{(r)\dagger}(l)$ is a creation operator for a loop of length l with + spins and satisfies with the corresponding annihilation operators $\Psi_+^{(r)}(l)$ the commutation relations

$$[\Psi_+^{(r)}(l), \Psi_+^{(r')\dagger}(l')] = \delta_{r,r'}\delta(l - l') \tag{6.2}$$

Hamiltonian H is a functional of $\Psi_+^{(r)}$, $\Psi_+^{(r)\dagger}$ and related to the loop equations. To determine its form we note that the existence of the limit (6.1) implies string field SDE

$$\lim_{D\to\infty} \frac{\partial}{\partial D} < 0|e^{-DH} \exp\{\int_0^\infty dl \sum_{r=1,2} J_+^{(r)}(l)\Psi_+^{(r)\dagger}(l)\}|0 >= 0. \tag{6.3}$$

This can be rewritten as a differential equation for Z_+

$$\hat{H}[J_+^{(r)}, D_+^{(r)}]Z_+[J_+^{(1)}, J_+^{(2)}] = 0, \tag{6.4}$$

where $\hat{H}$ is obtained from H by replacing $\Psi_+^{(r)}(l)$ and $\Psi_+^{(r)\dagger}(l)$ by $J_+^{(r)}(l)$ and $D_+^{(r)}(l)$, respectively, and interchanging the ordering of J's and D's.

In the case of pure gravity ($c = 0$), $\hat{H}$ is given by[6]

$$\hat{H} = \int_0^\infty dl J(l)\{lT(l) - \rho(l)\}, \tag{6.5}$$

where

$$T(l) = \int_0^l D(l')D(l - l') + g\int_0^\infty dl' J(l')D(l + l') \tag{6.6}$$

and $\rho(l)$ is a tadpole term. If the generating function $Z[J]$ satisfies the loop equation

$$lT(l)Z[J] = \rho(l)Z[J] \tag{6.7}$$

$Z[J]$ also satisfies SDE (6.4). The converse can also be proved [6].

Therefore a simple generalization to $c = 1/2$ string will be to take as $\hat{H}$ [7]

$$\hat{H} = \int_0^\infty dl J_+(l)\{l^2 W_+(l) - \rho(l)\} \tag{6.8}$$
$$+g\int_0^\infty dl_1 \int_0^\infty dl_2 l_1 l_2 (l_1 + l_2)\{aJ_+(l_1)J_+(l_2) + b\bar{J}_+(l_1)\bar{J}_+(l_2)\}T_+(l_1 + l_2),$$

where a, b are some constants not yet determined and the tadpole term is given by

$$\rho(l) = 32(6\delta''(l) - t\delta(l)). \tag{6.9}$$

$J_+(l)$ and $\bar{J}_+(l)$ are defined by

$$J_+ \equiv \frac{1}{2}(J_+^{(1)} + J_+^{(2)}), \quad \bar{J}_+ \equiv \frac{1}{2}(J_+^{(1)} - J_+^{(2)}). \tag{6.10}$$

The dimensions of $\Psi_+^{(r)}$, $\Psi_+^{(r)\dagger}$ are given by $L^{4/3}$, $L^{-7/3}$, respectively and that of g is $L^{-14/3}$, where L denotes the dimension of the length of a loop. Hence the dimension of $\hat{H}$ is $L^{-2/3}$ and we get the result $[D] \sim L^{2/3}$. However, for the reason which will be stated in the next few paragraphs we can not draw a definite conclusion about the dimension of geodesic distance D in this paper.[8] This construction of Hamiltonian can also be extended in an obvious way to the theory where $-$ loops are included. In (6.8) W_+ and T_+ have to be replaced by X_+ and U_+, respectively and terms with $+$ and $-$ interchanged should be added.

By construction our $Z_+[J_+^{(1)}, J_+^{(2)}]$ satisfies loop equations (2.9), (2.10) and we may stop at this point. But if our string field theory is to have geometrical meaning, each interaction in the Hamiltonian should give proper decomposition of world sheets into propagators, vertices and tadpoles. In other words, the Hamiltonian should provide us with Feynman rules for calculation of loop amplitudes with geodesic distances among loops fixed.

In [7] a consistency condition for string field Hamiltonian was proposed. Let us consider a world sheet with the topology of a cylinder. Suppose that the minimum geodesic distance of the two boundaries is D. We are able to compute an amplitude for such a geometry by sewing two transfer matrices by a disk. Starting from the two boundaries two loops propagate by splitting and disappearing and eventually two loops from the two boundaries meet at a point. Let the geodesic distance between this point and the two boundaries be D_1 and D_2 ($D_1 + D_2 = D$). After the two loops meet at a point, they

[7] Similar Hamiltonian is constructed by a different method in [24]. Main difference between our Hamiltonian and theirs is that their Hamiltonian does not contain terms corresponding to the singular terms of one- and two-loop amplitudes.

[8] For discussion of the dimension of geodesic distance D see [7], [8],[25],[26]

merge and the loop will keep splitting and disappearing to form a disk. In [7] it was proposed to require that the resulting amplitude should not depend on how the geodesic distance D is decomposed into two. This is a reasonable requirement if we are to compute such amplitudes in string field theory.

We performed such a consistency check of our Hamiltonian (6.8) and found the consistency condition in the above sense does not seem to be satisfied. The obstruction seems to lie in the fact that our loop equations contain extra degrees of freedom $\bar{J}_\pm = (J_\pm^{(1)} - J_\pm^{(2)})/2$ in addition to $J_\pm = (J_\pm^{(1)} + J_\pm^{(2)})/2$. In the next section we will construct loop equations for $J_\pm$ only.

7 Discussions

In this paper we derived the loop equations (5.8) in $c = \frac{1}{2}$ string, where Ising spins on the loops are all either up or down. These loop equations are equivalent to the W_3 constraints [2][3][14] which were derived in large-N matrix models and the consistency conditions of the loop equations yield an algebra which is a generalization of the W_3 algebra. Although our loop equations contain only the same information as the W_3 constraints, they are expected to provide us with a geometrical setting for constructing $c = 1/2$ SFT. Unfortunately the purpose of constructing SFT in sec 6 was not completely fulfilled. In this section we will construct loop equations for only $J_\pm = (J_\pm^{(1)} + J_\pm^{(2)})/2$ by eliminating $\bar{J}_\pm = (J_\pm^{(1)} - J_\pm^{(2)})/2$ from (5.8) and speculate on possible application of the result to SFT.

Let us first consider loop equations (2.9), (2.10) for + loops. Because we need loop equations only for loops corresponding to the source J_+, we rewrite $T_+(l)$ and $W_+(l)$ in terms of J_+, $\bar{J}_+$ and $D_+ \equiv D_+^{(1)} + D_+^{(2)}$, $\bar{D}_+ \equiv D_+^{(1)} - D_+^{(2)}$.

$$T_+ = \frac{1}{4} D_+ * D_+ - \frac{1}{4} \bar{D}_+ * \bar{D}_+ + g(lJ_+) \triangleleft D_+ + g(l\bar{J}_+) \triangleleft \bar{D}_+, \tag{7.1}$$

$$\begin{aligned} W_+ \;\; = \;\; & -3(D_+ * + 2g(lJ_+)\triangleleft)T_+ + (D_+ * + 3g(lJ_+)\triangleleft)^2 D_+ \\ & -3g(l\bar{J}_+) \triangleleft (D_+ * \bar{D}_+ - g(l\bar{J}_+) \triangleleft D_+ + g(lJ_+) \triangleleft \bar{D}_+) \\ & +3gD_+ * ((l\bar{J}_+) \triangleleft \bar{D}_+) + 3g^2 (lJ_+) \triangleleft ((l\bar{J}_+) \triangleleft \bar{D}_+). \end{aligned} \tag{7.2}$$

This result shows that we can eliminate $\bar{D}_+$ from the loop equations by setting $\bar{J}_+ = 0$. By using (7.2) and the equation in footnote 4, we obtain

$$\begin{aligned} & l^2 \{ (D_+ * + 3g(lJ_+)\triangleleft)^2 D_+ \}(l) Z_+ \\ & = -3gl^2 \frac{\partial}{\partial t} D_+(l) Z_+ + 32(6\delta''(l) - t\delta(l)) Z_+. \end{aligned} \tag{7.3}$$

This agrees with the result in [8], where the right-hand side in (7.3) was set to zero (denoted ≈ 0) as being terms either with a support at $l = 0$ or proportional to backgrounds.

Similarly we can derive loop equations for J_+ and J_- from (5.8). We obtain [9]

$$\begin{aligned}
& l^2\{D_+ * D_+ * D_+ + 3gD_+ * ((lJ_+) \triangleleft D_+) + 3g(lJ_+) \triangleleft (D_+ * D_+) \\
& -3g(lJ_-) * D_+ * D_+ - 3gD_+ * ((lJ_-) \triangleright D_-) - 3g(lJ_-) \triangleright (D_- * D_-) \\
& -9g^2(lJ_-) \triangleright ((lJ_-) \triangleleft D_-) + 9g^2(lJ_+) \triangleleft ((lJ_+) \triangleleft D_+) \\
& -9g^2(lJ_-) * ((lJ_+) \triangleleft D_+)\}(l)Z_{+-} \\
= \quad & 32\{6\delta''(l) - t\delta(l)\}Z_{+-} - 3gl^2 D_+(l)\frac{\partial}{\partial t}Z_{+-} + 9g^2 l^3 J_-(l)\frac{\partial}{\partial t}Z_{+-} \qquad (7.4)
\end{aligned}$$

and the counterpart obtained by interchanging + and − in (7.4). These are generalizations of (7.3) to + and − loops. Let us note that while $X_+(l)$ in (B.1) contain terms which are cubic in $J_\pm$'s, these terms are canceled altogether in the above equation.

This loop equation may serve as a starting point for construction of SFT for $c = \frac{1}{2}$ string. We can show that (7.4) can be decomposed into a set of loop equations.

$$\begin{aligned}
& \{D_+ * D_+ + 3g(lJ_+) \triangleleft D_+ - 3g(lJ_-) * D_+ - 3g(lJ_-) \triangleright D_- + \frac{\delta}{\delta K_+}\}Z_{+-} \approx 0, \\
& \{D_+ * \frac{\delta}{\delta K_+} + 3g(lJ_+) \triangleleft \frac{\delta}{\delta K_+} - 3g(lJ_-) \triangleright \frac{\delta}{\delta K_-}\}Z_{+-} \approx 0 \qquad (7.5)
\end{aligned}$$

Here $K_+(l)$, $K_-(l)$ are source functions for some auxiliary loop fields. It is understood that $K_\pm(l)$ are set to zero after differentiation in (7.5). These are generalization of the loop equations considered in [8].

$$\{\frac{\delta}{\delta J_0} * \frac{\delta}{\delta J_n} + (lJ_0) \triangleleft \frac{\delta}{\delta J_n} + \frac{\delta}{\delta J_{n+1}}\}Z|_{J_i=0} \approx 0 \quad (n = 0, 1, 2) \qquad (7.6)$$

Here J_0 is identical to J_+, while J_1 is a source for a + loop with a single operator $\mathcal{H}$ insertion, and J_2 that with double $\mathcal{H}$ insertion. Construction of SFT Hamiltonian based on loop equation (7.5) is under study.[27]

Whether loop equation (7.4) is compatible with that of Ishibashi and Kawai [7]

$$l\{D_+ * D_+ + D_+ \triangleright D_- + g(lJ_+) \triangleleft D_+\}(l)Z = 0 \qquad (7.7)$$

is an important problem. This is under investigation and we hope to report on this in the future [27]. If these loop equations are equivalent, the present

[9] The equation in footnote 6 is also used

work will provide us with a connection of Ishibashi-Kawai's SFT for $c = 1/2$ gravity to W_3 constraints. If not, $c = 1/2$ SFT has to be constructed from loop equations (7.4).

The structures of the loop equations (5.8) are quite interesting. For example the Virasoro constraint $U_+(l)$ is obtained by performing Bogoliubov transformation (5.5) on $T_+(l) - V_-(l)$. These mathematical structures may worth further investigation. It should also be pointed out that the Virasoro generators $U_\pm$ involve terms quadratic in J's and the W_3 currents $X_\pm$ contain cubic J terms.

We can extend the present analysis to $c = 1 - \frac{6}{m(m+1)}$ string. It is known that this theory can be constructively defined in terms of $(m-1)$-matrix-chain model and that the generating function of correlation functions in this theory satisfies W_m constraints. So corresponding to the n-th matrix M_n $(n = 1, \cdots, m-1)$ we need to introduce $m-1$ source functions $J_n^{(r)}(l) \quad (r = 1, \cdots, m-1)$. It will be straightforward to construct loop equations at least for two matter configurations on the loops corresponding to the ends M_1, M_{m-1} of the matrix chain.

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A Equivalence of Loop Eq (4.5) to the Virasoro Constraint (2.2)

In this appendix we will show that the loop equation (4.5) is equivalent to the Virasoro constraint (2.2).

The generating function Z_{+-} can be factorized into two parts.

$$Z_{+-} = Z_{+-}^{sing} Z_{+-}^{reg} \tag{A.1}$$

The singular part Z_{+-}^{sing} is an exponential of quadratic forms of J's and given

by

$$\ln Z_{+-}^{sing} = \frac{\sqrt{3}}{2\pi} g \int_0^\infty dl \int_0^\infty dl' \frac{l^{1/3} l'^{2/3}}{l+l'} \{J_+^{(1)}(l) J_+^{(2)}(l') + J_-^{(1)}(l) J_-^{(2)}(l')\}$$
$$- \frac{\sqrt{3}}{2\pi} g \int_0^\infty dl \fint_0^\infty dl' \frac{l^{1/3} l'^{2/3}}{l-l'} \{J_+^{(1)}(l) J_-^{(2)}(l') + J_-^{(1)}(l) J_+^{(2)}(l')\}$$
$$+ \frac{2^{4/3}}{3\Gamma(2/3)} \int_0^\infty dl (\frac{4}{3} l^{-7/3} - t l^{-1/3}) \{J_+^{(2)}(l) + J_-^{(2)}(l)\} \qquad (A.2)$$

Here the symbol $\fint$ means a principal value of an integral.

When $\tilde{D}_\pm^{(r)}(\zeta)$ operates on the regular part Z_{+-}^{reg}, it is assumed to have the following expansions.

$$\tilde{D}_+^{(r)}(\zeta) Z_{+-}^{reg} = g \sum_{n=0}^{\infty} \zeta^{-n-1-r/3} \frac{\partial}{\partial \mu_{3n+r}} Z_{+-}^{reg}, \qquad (A.3)$$
$$\tilde{D}_-^{(r)}(\zeta) Z_{+-}^{reg} = g \sum_{n=0}^{\infty} (-1)^{n+r+1} \zeta^{-n-1-r/3} \frac{\partial}{\partial \lambda_{3n+r}} Z_{+-}^{reg} \qquad (A.4)$$

Here μ_n and λ_n are source variables for scaling operators and defined by

$$\int_0^\infty dl J_+^{(r)}(l) l^{n+r/3} = \frac{1}{g} \Gamma(n+1+\frac{r}{3}) \mu_{3n+r}, \qquad (A.5)$$
$$\int_0^\infty dl J_-^{(r)}(l) l^{n+r/3} = \frac{1}{g} \Gamma(n+1+\frac{r}{3}) \lambda_{3n+r} (-1)^{n+r} \qquad (A.6)$$

By using these definitions we will rewrite the loop equation (4.5) into a differential equation for Z_{+-}^{reg}. After shifting the source variables according to

$$\mu_1 \to \mu_1 + (2)^{1/3} t, \qquad \mu_7 \to \mu_7 - \frac{3}{7} 2^{1/3},$$
$$\lambda_1 \to \lambda_1 + (2)^{1/3} t, \qquad \lambda_7 \to \lambda_7 - \frac{3}{7} 2^{1/3}, \qquad (A.7)$$

we obtain

$$\partial_\zeta \{ g^2 \sum_{n,m=0}^{\infty} \zeta^{-n-m-3} \frac{\partial}{\partial \mu_{3n+1}} \frac{\partial}{\partial \mu_{3m+2}}$$
$$+ g \sum_{r=1,2} \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \zeta^{m-n-2} (m + \frac{r}{3}) (\mu_{3m+r} + \lambda_{3m+r}) \frac{\partial}{\partial \mu_{3n+r}}$$
$$+ g \sum_{r=1,2} \sum_{n=0}^{\infty} \sum_{m=n+3}^{\infty} \zeta^{m-n-2} (m + \frac{r}{3}) \lambda_{3m+r} (\frac{\partial}{\partial \mu_{3n+r}} - \frac{\partial}{\partial \lambda_{3n+r}})$$
$$+ \frac{g}{9} \frac{1}{\zeta^2} + \frac{2}{9} \frac{1}{\zeta} (\mu_1 + \lambda_1)(\mu_2 + \lambda_2) \} Z_{+-}^{reg} = 0 \qquad (A.8)$$

Let us note that the third term is composed of positive powers of ζ. This term, however, shows that Z_{+-}^{reg} depends on μ and λ only through the combination $\mu_n + \lambda_n$. Then (A.8) reduces to the Virasoro constraint (2.2) on $\tau(\mu + \lambda) = Z_{+-}^{reg}$.

B 'Extended' W_3 Algebra

The operator $X_+(l)$ defined in sec 5 is given by

$$
\begin{aligned}
X_+(l) \;=\; & \sum_{r=1}^{2}\{D_+^{(r)} * D_+^{(r)} * D_+^{(r)} + 3g(lJ_+^{(3-r)}) \triangleleft (D_+^{(r)} * D_+^{(r)}) \\
& -\frac{3}{2}g(lJ_-^{(3-r)}) * D_+^{(r)} * D_+^{(r)} - 3g((lJ_-^{(r)}) \triangleright D_-^{(3-r)}) \triangleright D_-^{(3-r)} \\
& +3g^2(lJ_+^{(3-r)}) \triangleleft ((lJ_+^{(3-r)}) \triangleleft D_+^{(r)}) + \frac{3}{4}g^2(lJ_-^{(3-r)}) * (lJ_-^{(3-r)}) * D_+^{(r)} \\
& -3g^2(lJ_+^{(3-r)}) \triangleleft ((lJ_-^{(3-r)}) * D_+^{(r)}) + 3g^2((lJ_-^{(3-r)}) \triangleright (lJ_+^{(3-r)})) \triangleright D_-^{(r)} \\
& -3g^2((lJ_-^{(r)}) * (lJ_-^{(r)})) \triangleright D_-^{(3-r)} - \frac{9}{8}g^3(lJ_-^{(r)}) * (lJ_-^{(r)}) * (lJ_-^{(r)}) \\
& -\frac{9}{4}g^3(lJ_+^{(r)}) \triangleleft ((lJ_+^{(r)}) \triangleleft (lJ_-^{(r)})) + \frac{9}{4}g^3(lJ_+^{(r)}) \triangleleft ((lJ_-^{(r)}) * (lJ_-^{(r)}))\}(l)
\end{aligned}
\tag{B.1}
$$

In the remaining part of this Appendix we will show the algebra that $U_\pm$ and $X_\pm$ generate. First of all $T_\pm$, $V_\pm$, $W_\pm$ and $Y_\pm$ generate the following algebra. Some formulae which are not presented here can be found in (2.19)$\sim$(2.21), (2.23), (2.24). The index $\pm$ will be suppressed here.

$$[T(l), Y(l')] = g(2l + l')W(l - l')\theta(l - l') + g(2l + l')Y(l' - l)\theta(l' - l), \tag{B.2}$$

$$[V(l), W(l')] = -g(2l + l')Y(l - l')\theta(l - l') - 2(2l' + l)W(l' - l)\theta(l' - l), \tag{B.3}$$

$$[V(l), Y(l')] = -g(2l - l')Y(l + l'), \tag{B.4}$$

$$
\begin{aligned}
[Y(l), Y(l')] \;=\; & -9g(l - l')(V * V)(l + l') - 18g(l - l')(V \triangleright T)(l + l') \\
& +\frac{3}{2}g^2(l - l')(l^2 + 4ll' + l'^2)V(l + l'),
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
[Y(l), W(l')] \;=\; & \theta(l' - l)\{-9g(l + l')(T * T)(l' - l) \\
& -18g(l + l')(V \triangleleft T)(l' - l) + \frac{3}{2}g^2(l + l')(l^2 - 4ll' + l'^2)T(l' - l)\} \\
& +\theta(l - l')\{-9g(l + l')(V * V)(l - l') - 18g(l + l')(V \triangleright T)(l - l') \\
& +\frac{3}{2}g^2(l + l')(l^2 - 4ll' + l'^2)V(l - l')\}
\end{aligned}
\tag{B.6}
$$

By using these relations we can derive the algebra of $\hat{U}_\pm = T_\pm - V_\mp$ and $\hat{X}_\pm = W_\pm - Y_\mp$.

$$[\hat{U}_\pm(l), \hat{X}_\pm(l')] = g(2l - l')\hat{X}_\pm(l + l'), \tag{B.7}$$

$$[\hat{U}_\pm(l), \hat{X}_\mp(l')] = -g(2l + l')\hat{X}_\pm(l - l')\theta(l - l') + g(2l + l')\hat{X}_\mp(l' - l)\theta(l' - l), \tag{B.8}$$

$$\begin{aligned}
[\hat{X}_\pm(l), \hat{X}_\pm(l')] &= -\frac{3}{2}g^2(l - l')(l^2 + 4ll' + l'^2)\hat{U}_\pm(l + l') \\
&\quad +9g(l - l')\{T_\pm * \hat{U}_\pm + V_\mp * \hat{U}_\pm + 2V_\pm \triangleleft \hat{U}_\pm \\
&\quad -2V_\mp \triangleright \hat{U}_\mp\}(l + l'),
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
[\hat{X}_+(l), \hat{X}_-(l')] &= \theta(l - l')[\frac{3}{2}g^2(l + l')(l^2 - 4ll' + l'^2)\hat{U}_+(l - l') \\
&\quad -9g(l + l')\{T_+ * \hat{U}_+ + V_- * \hat{U}_+ + 2V_+ \triangleleft \hat{U}_+ \\
&\quad -2V_- \triangleright \hat{U}_-\}(l - l')] \\
&\quad -(l \leftrightarrow l', + \leftrightarrow -)
\end{aligned} \tag{B.10}$$

Other commutation relations are given in (5.2) and (5.3). The algebra of $U_\pm$ and $X_\pm$ can be derived from the above by transformation (5.5).

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